PP36672. Proposed by Mihaly Bencze.

If $a, b > 0$ then

$$\frac{a^2+16ab+80b^2}{16(a+4b)}+\frac{b^2+16ab+80a^2}{16(b+4a)}+\sqrt{2}\left(\frac{a^2}{\sqrt{b^2+16a^2}}+\frac{b^2}{\sqrt{a^2+16b^2}}\right) \geq \frac{3(a+b)}{2}.$$

Solution by Arkady Alt, San Jose, California, USA.

First we will prove that for any $x, y > 0$ holds inequality

(1) $$\frac{x^2+16xy+80y^2}{16(x+4y)}+\frac{\sqrt{2}\,y^2}{\sqrt{x^2+16y^2}} \geq \frac{3y}{2}.$$

Proof.

Since $x+4y \leq \sqrt{2(x^2+16y^2)} \iff (x-4y)^2 \geq 0$ then $\dfrac{x^2+16xy+80y^2}{16(x+4y)} =$

$\dfrac{x^2+16y^2}{16(x+4y)}+y \geq \dfrac{x^2+16y^2}{16\sqrt{2(x^2+16y^2)}}+y = \dfrac{\sqrt{x^2+16y^2}}{16\sqrt{2}}+y$ and, therefore,

by AM-GM Inequality

$$\frac{x^2+16xy+80y^2}{16(x+4y)}+\frac{\sqrt{2}\,y^2}{\sqrt{x^2+16y^2}} \geq \frac{\sqrt{x^2+16y^2}}{16\sqrt{2}}+\frac{\sqrt{2}\,y^2}{\sqrt{x^2+16y^2}}+y \geq$$

$$2\sqrt{\frac{\sqrt{x^2+16y^2}}{16\sqrt{2}}\cdot\frac{\sqrt{2}\,y^2}{\sqrt{x^2+16y^2}}}+y = 2\sqrt{\frac{y^2}{16}}+y = \frac{y}{2}+y = \frac{3y}{2}.$$

Applying (**1**) to $(x, y) = (a, b)$ and $(x, y) = (b, a)$ we obtain

$$\frac{a^2+16ab+80b^2}{16(a+4b)}+\frac{b^2+16ab+80a^2}{16(b+4a)}+\sqrt{2}\left(\frac{a^2}{\sqrt{b^2+16a^2}}+\frac{b^2}{\sqrt{a^2+16b^2}}\right) =$$

$$\left(\frac{a^2+16ab+80b^2}{16(a+4b)}+\frac{\sqrt{2}\,b^2}{\sqrt{a^2+16b^2}}\right)+\left(\frac{b^2+16ab+80a^2}{16(b+4a)}+\frac{\sqrt{2}\,a^2}{\sqrt{b^2+16a^2}}\right) \geq$$

$$\frac{3b}{2}+\frac{3b}{2} = \frac{3(a+b)}{2}.$$